

# **A MATHEMATICAL PROBLEM-SOLVING EXERCISE BY IN-SERVICE TEACHERS IN A CONTINUOUS TEACHING DEVELOPMENT CONTEXT.**

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*This paper is a reflective description of a problem-solving exercise by 40 in-service teachers in a mathematics continuous development workshop. The exercise was aimed at assessing the teachers' subject content knowledge and problem-solving strategies when given a non-routine trigonometric identity to prove. The teachers were divided into three groups and given a time of approximately one hour to prove the identity. At the end of the allotted time, the teachers presented their solution. The solution reveals that teachers were able to solve the problem after following the problem-solving steps proposed by Polya and Schoenfeld's criteria, whilst one group did not complete the process.*

## **INTRODUCTION**

This paper reflects on a problem-solving exercise by in-service teachers participating in a mathematics continuous professional development project, the Local Evidence Driven Improvement in Mathematics Teaching and Learning Initiative (LEDIMTALI) led by a higher education institution in the Western Cape Province of South Africa. In this project mathematics teacher educators, mathematicians, mathematics teachers and mathematics curriculum advisors work collectively and collaboratively to facilitate quality teaching of mathematics. The initiative is premised on the belief that such collective and collaborative work can cascade to learners achieving at their highest potential in mathematics.

The LEDIMTALI project which started in earnest in 2012 endeavours to establish and develop a community of goal driven mathematics practitioners by:

- (a) Providing opportunities for reflections on classroom-based teaching of mathematics.
- (b) Designing and developing strategies to enhance ways of teaching mathematics based on results forthcoming from these considered reflections.
- (c) Supporting teaching of mathematics through coaching, training and co-teaching.
- (d) Providing resources to enhance both the teaching and learning of mathematics.
- (e) Developing opportunities for the enhancement and understanding of the appropriate mathematical knowledge that underlies the teaching of school mathematics.

- (f) Developing respectful ways of working amongst and between mathematics educators, mathematics, mathematics teachers and mathematics curriculum advisors for the enhancement of teaching mathematics. (Julie, 2011, p.1).

To achieve the goals articulated by the LEDIMTALI document, regular three hour workshops and are held and attended by all the stakeholders.

In addition bi-monthly teacher institutes are also organised as residential workshops whereby all participants stay over for a weekend for intensive discussion of the mathematics curriculum, pedagogy and general mathematical engagements.

Activities which generally take place at these institutes include the following:

- Reports and reflections on teaching experiments, workshops and school teaching experiences
- The teaching of mathematical concepts and design of lessons.
- Setting of end-of-year examination for grades 10 and 11
- Engaging with mathematical content on request by teachers or from proposal by teacher educators.

It is in the context of “Developing opportunities for the enhancement and understanding of the appropriate mathematical knowledge that underlies the teaching of school mathematics” (Julie, 2011) that one of the institutes dedicated a two hour session on problem-solving on the 18<sup>th</sup> October 2013 in . One of the problems in this workshop involved the proof of a trigonometric identity. This paper analyses and describes the strategies which teachers employed in proving the identity.

## **LITERATURE REVIEW**

The basic assumption for these kinds of mathematical engagement sessions in LEDIMTALI is the imperative of the enhancement of teacher knowledge of mathematical content on the understanding that this would have an effect on the teachers’ ability to teach the subject to their learners (Jadama, 2014). LEDIMTALI assumes that as part of continuous professional development and in addition to the discussion on curriculum issues, regular mathematical engagement should occur so that teachers’ knowledge should be constantly sharpened. As Shulman (1987) puts it “...Thus teaching necessarily begins with a teacher’s understanding of what is to be learned and how it is to be taught.” (p7). This is informed by the notion of teachers’ subject content knowledge which has received a lot of attention from mathematics educators recently. Much of the writing on this issue derive from the seminal work of Shulman (1987) who propounded a taxonomy of what he proposes should be a knowledge base of what teachers should possess. He proposes seven categories of teacher knowledge inter alia, content knowledge, pedagogical content knowledge and curriculum knowledge. He refers to subject content knowledge as “- the knowledge, understanding and skill, and disposition that are to be learned by school children.” (Shulman, 1987. p8).

In addition Shulman posits that the teacher should know more than the basic content namely how the subject matter is structured and organised and the kinds of questions that are pursued in the particular field of study. This is important because the teacher is primarily responsible for ensuring that students understand the subject matter. Most authors concur on the importance of the teachers' content knowledge as an important prerequisite for effective teaching (Fennema & Franke, 1992; Lucus, 2006; Ma, 1999; Norman, 1992; Stein, Baxter & Leinhardt, 1990)

This paper reflects on an exercise which gave teachers an opportunity to demonstrate their knowledge of trigonometry by participating in a trigonometric proof. This we regard as part of a process of problem-solving.

Problem-solving has been proposed in the last four or more decades as an objective for mathematics teaching and learning by many mathematical associations throughout the world. The American National Council of Teachers of Mathematics for instance in their *Principles and Standards for School Mathematics* (NCTM, 2000) state that problem-solving should not only be a goal for learning mathematics but should also be a means for doing so. The NCTM states that:

Problem-solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will develop new mathematical understanding. ...Student should have frequent opportunities to formulate, grapple with and solve complex problems that require significant amount of effort and should then be encouraged to reflect on their thinking (NCTM, 2000, 52).

In South Africa, the new Curriculum and Assessment Policy Statement (CAPS) of the national Department of Basic Education (DBE) states that:

Problem-solving and cognitive development should be central to all mathematics teaching. Learning procedures and proofs without a good understanding of why they are important will leave learners ill-equipped to use their knowledge in later life. (p.8)

Many mathematics educators that we have engaged with propose that problem-solving should not only be seen as an activity that occurs in the classroom but should extend to the application of mathematics to real life outside the classroom. We believe that teachers themselves should experience problem-solving in the same ways that learners should.

Problem-solving can be regarded as:

...the means by which individuals take the skills and understandings they have developed previously and apply them to unfamiliar situations. The process begins with the initial confrontation of the problem and continues until an answer has been obtained and the learner has examined the solution. (Krulik, Rudnick & Milou, 2003:93).

Reys, Suydam & Lindquist (1984) give an analogy of problem-solving as “...involving a situation in which a person wants something but does not know immediately what to do to get it.” (p. 27) hence if it is immediately clear what to do to obtain what one wants, then there is no problem.

In essence problem-solving is not a simple matter of routine problems using readily available algorithms. One needs to think a while about what strategies to use to solve the problem. Hence cognition and higher order thinking are imperatives in the solution of problems. Higher order or critical thinking is an important element in enabling one to find a way of analysing the problem to find a solution path (Brumbagh & Rock, 2006).

Our stance on problem-solving is to recognise that there is a multiplicity of conceptions on it and various approaches in concurrence with Schoenfeld’s (1992) view that there exist contending and contradictory views of problem-solving. We accept his view that problem-solving should “train students to ‘think’ creatively and/or develop their problem-solving ability (with a focus on heuristic strategies)” (Schoenfeld, 1992, 10)

Polya (1945), in his seminal work, *How to solve it*, proposes four steps which one has to go through in the solution of a problem.

These steps are:

1. *Understanding the Problem:*

Understanding the problem ensures that the learner must first understand the verbal articulation of the problem and what it entails. This means that the student should be able to identify what is given and what is required.

2. *Devising a plan:*

This means that the learner must use what is given and think of possible strategies to implement towards the goal of solving the problem. The strategy may not be easy. Polya (1945) states that this “may be long and tortuous.” (p.8)

3. *Implementing the plan:*

This is self-explanatory. It means that the student having devised a plausible plan may then execute the solution plan.

4. *Looking back (Reflection):*

Looking back means that having solved the problem, the student has to look back to see if the solution is reasonable and correct because errors might have crept into the solution process.

## RESEARCH DESIGN

A qualitative research approach embedded in a constructivist paradigm was used in this study.

The data for this paper was obtained from a problem-solving exercise by 30 respondents comprising mathematics teachers and mathematics curriculum advisors held at our LEDIMTALI workshops in 2013. The participants were divided into groups and given approximately an hour to solve some trigonometric and financial mathematics problems and then present the group solution to all present. Subsequently, group discussed solutions were presented and motivated by representatives of each group. Each of these solutions was reflected upon and debated in terms of its mathematical correctness by all participants. The whole process from engaging with problems to exploring solutions up to their presentations and debates were video recorded.

This paper focuses on the following trigonometric problem because it involves proof which is an area of interest to the authors. The problem selected from Mathematical Digest.

Prove that in  $\triangle ABC$ ,  $\frac{\tan B}{\tan C} = \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$  (Webb, Hardie, Allison, Brummer, Gilmour & Conradie, 1987)

It should be noted that in trigonometry, identities are proved in various ways. One method is to take the left hand side (or the right hand side) of the identity and simplify until it is identical to the right hand side (or the left hand side). Another method is to reduce both sides to a common expression or identity. It is important to note that during the proving process relevant trigonometric identities may be used to convert or simplify some of the identities to what is required to be proved. Sometimes reducing all or some of the identities or trigonometric ratios to sines and cosines would be an alternative way of proving.

In this paper we discuss three different ways of proving the identity presented by teachers in the workshop.

## CASE 1

NO. 2.

$$\frac{\tan B}{\tan C} = \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$$

$$\frac{\sin B}{\cos B} \div \frac{\sin C}{\cos C} = \frac{2ab \cos C}{2ac \cos B} \quad \left( \begin{array}{l} \text{By} \\ \text{Cosine} \\ \text{Rule} \end{array} \right)$$

$$\frac{\sin B}{\cos B} \times \frac{\cos C}{\sin C} = \frac{b \cos C}{c \cos B}$$

Then By sin Rule:  $\frac{b}{\sin B} = \frac{c}{\sin C}$

$$\therefore \frac{b}{c} = \frac{\sin B}{\sin C}$$

$$\frac{\sin B}{\sin C} \cdot \frac{\cos C}{\cos B} = \frac{b \cos C}{c \cos B}$$

Looking at this solution one can observe that the teachers attempted to use the method of reducing the right hand side (RHS) and the left hand side simultaneously to the same expression, in this case,  $\frac{b \cos C}{c \cos B}$ . What we find in this solution is that the teachers invoked the sine rule, the cosine rule and the quotient identities for the tangent ratios. Although the structure of the proof is not in the conventional form, the group has brought to the proof salient ideas. For example it seems that the teachers have realised that they can transform the numerator and denominator of the expression on the RHS to incorporate a trigonometric ratio by applying the cosine rule to  $\triangle ABC$ .

In particular they have used the cosine rule to express the numerator  $a^2 + b^2 - c^2$  as  $2ab \cos C$  and the denominator  $a^2 + c^2 - b^2$  as  $2ac \cos B$  as shown in line 2 of the proof. Thereafter through algebraic simplification (i.e. cancellation of like factors), they reduced the quotient  $\frac{2ab \cos C}{2ac \cos B}$  to  $\frac{b \cos C}{c \cos B}$ .

It seems that group manipulated left hand side in the following sequence until they arrived at  $\frac{b\cos C}{c\cos B}$ :

- They use the tan quotient identity to express  $\tan B$  as  $\frac{\sin B}{\cos B}$  and  $\tan C$  as  $\frac{\sin C}{\cos C}$
- In step 3, it seems that the group realized that they could express  $\frac{\sin B}{\cos B} \times \frac{\cos C}{\sin C}$  as  $\frac{\sin B}{\sin C} \times \frac{\cos C}{\cos B}$
- Thereafter tried to show by using the sine rule that  $\frac{\sin B}{\sin C} = \frac{b}{c}$ .
- The teachers then substituted  $\frac{b}{c}$  for  $\frac{\sin B}{\sin C}$  with respect to the expression on LHS of the identity in step3 to arrive at  $\frac{b\cos C}{c\cos B}$ , implying that the LHS= RHS, and thus completing the proof.

This solution was debated and considered correct except the expression of the solution with the equality sign from the very first statement indicating an assumption of the veracity of the identity at the commencement of the proof. A proper and acceptable proof in this instance would have been to take the right and left hand sides separately until each side is simplified to reach the result of  $\frac{b\cos C}{c\cos B}$ . However, we find that the group did not state that none of angles B and C can be  $90^\circ$  for a given  $\Delta ABC$ .

## CASE 2

The second group presented the following proof:

$$\begin{aligned}\tan B &= \frac{\sin B}{\cos B} = \frac{b \cdot \sin C}{c} \times \frac{2ac}{a^2 + c^2 - b^2} \\ &= \frac{2ab \sin C}{a^2 + c^2 - b^2} \\ \tan C &= \frac{\sin C}{\cos C} = \frac{c \cdot \sin B}{b} \times \frac{2ab}{a^2 + b^2 - c^2} \\ &= \frac{2ac \sin B}{a^2 + b^2 - c^2} \\ \frac{\tan B}{\tan C} &= \frac{2ab \sin C}{a^2 + c^2 - b^2} \times \frac{a^2 + b^2 - c^2}{2ac \sin B} \\ &= \frac{b \cdot \sin C (a^2 + b^2 - c^2)}{c \cdot \sin B (a^2 + c^2 - b^2)} \\ \frac{\sin A}{a} &= \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \sin B = \frac{b \cdot \sin C}{c} \\ \sin A &= \frac{a \sin B}{b} \quad \text{and} \quad \sin A = \frac{a \sin C}{c} \\ &= \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}\end{aligned}$$

The strategy adopted by this group was to work with the LHS of the identity and express  $\tan B$  and  $\tan C$  in terms of the sines and cosines of  $\hat{B}$  and  $\hat{C}$ . Now

$\tan B = \frac{\sin B}{\cos B}$  by using quotient identities for the tangent. Similarly  $\tan C$  is expressed in terms of the sine and cosine as  $\tan C = \frac{\sin C}{\cos C}$ .



The group now proceeds to use a combination of the sine and cosine formula for the expression  $\frac{\sin B}{\cos B}$ . Using the sine formula for the angles  $B$  and  $C$ , we have that

$\sin B = \frac{b \sin C}{c}$  and then using the cosine formula with angle  $B$  as a reference angle we have that  $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$ .

Through using algebraic manipulation, expressed  $\frac{\sin B}{\cos B}$  was written in the form  $\frac{b \sin C}{c} \times \frac{2ac}{a^2 + c^2 - b^2} = \frac{2ab \sin C}{a^2 + c^2 - b^2}$ . Similarly  $\tan C$  was expressed as  $\frac{2ab \sin B}{a^2 + b^2 - c^2}$ .

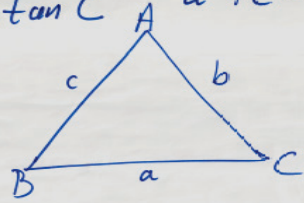
Thereafter, the group expressed  $\frac{\tan B}{\tan C} = \frac{2ab \sin C}{a^2 + c^2 - b^2} \times \frac{a^2 + b^2 - c^2}{2ac \sin B}$

Through cancellation of common factor 'a' in both the numerator and denominator, and having insight that  $b \sin C = C \sin B$  through using the sine rule, the group proceeded to write the express  $\frac{\tan B}{\tan C}$  as  $\frac{b \sin C (a^2 + b^2 - c^2)}{c \sin B (a^2 + c^2 - b^2)}$ . On the grounds that  $b \sin C = C \sin B$ , the group cancelled  $b \sin C$  in the numerator with  $C \sin B$  in the denominator. Hence, the group has shown that  $\frac{\tan B}{\tan C} = \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$ . However, we find that the group did not state that none of angles  $B$  and  $C$  can be  $90^\circ$  for a given  $\triangle ABC$ .

### Case 3

GRADES 10-12: PROBLEM 2

① Prove that in  $\triangle ABC$

$$\frac{\tan B}{\tan C} = \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$$


Cos Rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Sin Rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\sin B = \frac{b \sin C}{c} \quad \sin C = \frac{c \sin B}{b}$$

The last group presented the above incomplete proof. As can be observed above, it is clear that the group was able to identify the RHS as a quotient of expressions of the cosine formula. They began to express all sides of the triangle in terms of the cosine formula. They then made the cosines of angles B and C which are angles of the tangent quotient in the RHS of the identity to be proved, It seems that they were able to see the link between the tangents and the quotient ratios because they began to write down the sine formula. One hopes that if they had more time, they would be able to link the sine and sine formula like the first group and follow up to the proof.

## DISCUSSION

The proofs offered by both groups, demonstrate that they were able to read and understand the problem, and apply their minds to distil a strategy to solve the problem. Hence in this sense, it seems as if they followed Polya's problem-solving steps of initially reading to understand the problem, planning a strategy to tackle the problem and then solving the problem with a review at the end to check for correctness. Group 1 used the strategy of expressing the LHS and RHS to a common trig expression,  $\frac{b\cos C}{c\cos B}$ , through using quotient identities and the cosine formula coupled with algebraic manipulations. Group 2, started with less complex side (i.e. the LHS in this instance) and by using quotient identities, sine and cosine formula showed that the LHS = RHS. Unfortunately the third group did not finish the solution. One wonders why this group ran out of time compared to the other two groups. It might be a case of less experience teachers who did not receive assistance from their more experienced colleagues.

However, none of the groups took the more complex side (which is the RHS in this instance)  $\frac{a^2+b^2-c^2}{a^2+c^2-b^2}$ , and simplified to the LHS.

Referring to the "bottom lines" of success as articulated by Schoenfeld (2012), we can state that the teachers who are the subject of this paper were successful in as far as they were able to complete and present their solution, except of course the group who did not complete. The strategies employed by the teachers were clear and unambiguous. Unfortunately the teachers were not given time for metacognition or what is referred to by Schoenfeld (2012) as monitoring and self-regulation nor were they given time to discuss their experiences. Maybe it is the fault of the workshop facilitators.

It would also be interesting to see if one could construct a perpendicular from one vertex of the triangle to see if the usual right triangle definition of a tangent for the two angles B and C would yield the required identity. This exercise will also be followed up by interviews with some of the teachers to get a sense of their feeling about the process and their experiences.

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